Ultraviolet finiteness of Chiral Perturbation Theory for two-dimensional Quantum Electrodynamics

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Abstract

We consider the perturbation theory in the fermion mass (chiral perturbation theory) for the two-dimensional quantum electrodynamics. With this aim, we rewrite the theory in the equivalent bosonic form in which the interaction is exponential and the fermion mass becomes the coupling constant. We reformulate the bosonic perturbation theory in the superpropagator language and analyze its ultraviolet behavior. We show that the boson Green's functions without vacuum loops remain finite in all orders of the perturbation theory in the fermion mass.

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1. Introduction

Because of the difficulties that arise in analyzing gauge theories, it is interesting to investigate simple models that admit a nonperturbative description and, in particular, to investigate infinite perturbation theories (PT) in all orders. One such model is the two-dimensional quantum electrodynamics (2D-QED) with a nonzero fermion mass. This model, also called the massive Schwinger model, is described by the Lagrangian density

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^{\mu}D_{\mu} - M)\Psi, \tag{1}$$

where $\mu, \nu = 0, 1,$ $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},$ $D_{\mu} = \partial_{\mu} - ie_{\mathrm{el}}A_{\mu},$ A_{μ} is the vector potential of the electromagnetic field, $\Psi = \binom{\psi_-}{\psi_+},$ $\bar{\Psi} = \Psi^{+}\gamma^{0}$ is the field of the fermion with the mass $M,$ e_{el} is the analogue of the electron charge, and the matrices γ^{μ} can be taken in the form

$$\gamma^{0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^{1} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2}$$

It is known that this model is exactly solvable [1] in the case of a vanishing fermion mass M. In the case of $M \neq 0$, we can therefore consider the so-called chiral PT, i.e., the PT in the fermion mass M [2]. This is interesting because such a PT corresponds to an approach that is nonperturbative from the viewpoint of the usual PT. Indeed, the actual expansion is in the dimensionless ratio M/e_{el} (the coupling constant e_{el} has the dimension of mass for a two-dimensional theory). We can therefore consider the chiral PT as an expansion in the parameter $1/e_{\mathrm{el}}$, which is valid in the domain of large e_{el}.

In gauge theories with space-time dimensionality greater than two, the vanishing fermion mass does not suffice for making the model exactly solvable; therefore, the PT in the parameter M cannot be constructed. One can nevertheless try to consider the expansion in the parameter $1/e_{\mathrm{el}}$ in such theories, passing to the "dual" form of the model. We can study the features of such expansions using the chiral PT for 2D-QED, which provides a simple example of it.

One more domain of application of the chiral PT for 2D-QED is constructing the light-front (LF) Hamiltonian for such a model and studying the properties of the such approach on this example. The LF Hamiltonian approach [3] is completely nonperturbative (i.e., the corresponding Schrodinger equation can be solved at arbitrary values of the coupling constant), while the description of the physical vacuum state becomes trivial in this approach [4]. But because of the breaking of Lorentz invariance and the presence of singularities specific to the LF coordinates at the zero value of the lightlike momentum $p_{-} = (p_0 - p_3)/\sqrt{2}$ [5–7], the theory generated by the LF Hamiltonian can be nonequivalent to the initial Lorentz-covariant theory [8, 9]. In this case, restoring the equivalence requires adding coun-terterms to the LF Hamiltonian. The form of these counterterms can be found either by using the method of approximate limiting transition to the LF [9–11] or by comparing the contributions in all orders of the PT in the coupling constant for the theory generated by the LF Hamiltonian and for the Lorentz-covariant theory [12–14]. But the latter method cannot be applied to 2D-QED, because the PT in the coupling constant e_{el} fails in this model due to infrared divergences. Nevertheless, it is possible to construct the LF Hamiltonian for 2D-QED using the chiral PT [15–17].

The chiral PT for 2D-QED can be conveniently constructed by passing to the equivalent formulation of the theory in the bosonic variables using the bosonization procedure [18, 19, 9].

The corresponding bosonic Lagrangian is

$$L = \frac{1}{8\pi}\left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right) + \frac{\gamma}{2}e^{i\theta} : e^{i\varphi} : + \frac{\gamma}{2}e^{-i\theta} : e^{-i\varphi} :, \quad \gamma = \frac{Mme^C}{2\pi}, \quad m = \frac{e_{\rm el}}{\sqrt{\pi}}, \tag{3}$$

where $C = 0.577216$ is the Euler constant, the parameter θ is responsible for the choice of the instantonic θ- vacuum in the 2D-QED [18, 19, 2] and the normal-ordering symbol means that the PT in γ does not contain diagrams with loops containing only one vertex (this corresponds to the standard normal-ordering symbol in the Hamiltonian).

This Lagrangian describes the theory of a massive scalar field with an exponential interaction. The fermion mass M then plays the role of the coupling constant, and the chiral PT for 2D-QED therefore coincides with the standard PT for bosonic theory (3). The propagator is simple in this PT (it differs from the standard scalar field propagator only by a numerical factor), but Lagrangian (3) contains a nonpolynomial interaction. In each order of the PT in M, we therefore have an infinite number of diagrams. Each separate diagram converges, but it turns out that particular infinite sums of diagrams may diverge in a given order of the PT in M, and this divergence manifests the ultraviolet (UV) nature [2, 15]. The question of the presence of UV divergences in the PT in M for bosonic Green's functions is therefore nontrivial, although two-dimensional models are usually free of UV divergences. We note that this problem was not been completely solved [2, 20].

In this paper, we prove the absence of UV divergences in all orders of the PT in γ (and hence in M) for the Green's functions without vacuum loops in bosonic theory (3). We understand the absence of UV divergences to mean the finiteness of the result in the limit of removed UV regularization (we need an intermediate UV regularization because of the absence of absolute convergence of the sum of diagrams in a given order of the PT expansion in γ). The result is proved to be finite because all divergences cancel when all the contributions of the given order in γ are summed in any given Green's function.

2. Absence of surface UV divergences in Green's functions of orders higher than the second in γ

We consider the structure of diagrams of the Feynman PT in γ for Lagrangian (3). There are two types of vertices with j external lines ($j = 0, 1, 2, \ldots$) generated by two interaction terms in (3). The factors

$$i^{j+1}\,\frac{\gamma}{2}e^{i\theta} \qquad \text{and} \qquad i^{-j+1}\,\frac{\gamma}{2}e^{-i\theta} \tag{4}$$

correspond to these vertices respectively for the first and second types of interaction. Vertices with $j = 0$, i.e., without lines, must be considered subdiagrams of a nonconnected diagram. It is convenient to relate the part $i^{\pm j}$ of the vertex factors to the lines that are external with respect to a vertex (we set $\pm i$ to each line); the vertex factors then become

$$i\,\frac{\gamma}{2}e^{i\theta} \qquad \text{and} \qquad i\,\frac{\gamma}{2}e^{-i\theta}. \tag{5}$$

The propagator $\Delta(x) = \langle 0|T\varphi(x)\,\varphi(0)|0\rangle$, where the field $\varphi(x)$ corresponds to the free theory described by the Lagrangian (3) with $\gamma = 0$, is

$$\Delta(x) = \int d^2k\; e^{ikx}\Delta(k), \qquad \Delta(k) = \frac{i}{\pi}\frac{1}{(k^2 - m^2 + i0)}, \tag{6}$$

where $d^2k = dk_0 dk_1$, $kx = k_0 x^0 + k_1 x^1$.

Because the theory with Lagrangian (3) contains exponential interaction terms, we can reformulate the PT in γ in the superpropagator language, i.e., in terms of sums of contributions of the same order in γ corresponding to all possible variants of joining a pair of vertices by different numbers of propagators [2, 16]. For a pair of vertices of different types, the superpropagator is

$$\sum_{m=0}^{\infty} \frac{1}{m!} \Delta(x)^m = e^{\Delta(x)}, \tag{7}$$

while for a pair of vertices of the same type, it is

$$\sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \Delta(x)^m = e^{-\Delta(x)}. \tag{8}$$

In formulas (7) and (8), the factor $1/m!$ is the symmetry coefficient, which corresponds to m parallel lines. The parts of vertex factors (4) joined to the lines (see the reasoning after that formula) are taken into account. In such an approach, the sum of all standard (including nonconnected) diagrams with the given number of vertices and the given way of attaching the external line couplings is described by a single diagram in which each pair of vertices is joined by the corresponding superpropagator (the connectedness is always understood as the standard connectedness). For the external lines, we set the usual propagators (6) with the additional factor $\pm i$, which is a part of the vertex factor related to a line.

We now perform the Wick rotation, thus passing to the Euclidean space, and introduce an intermediate UV regularization for propagator (6) (e.g., by the higher-derivative method).

We consider the (infinite) sum S'_n of all the diagrams (including nonconnected) of the order n in the PT in γ for which the number of external lines coupled to each vertex is fixed and the type of each vertex is also fixed. In terms of superpropagators, this sum is a single diagram with each pair of vertices joined by the corresponding superpropagator (7) or (8).

We let S_n denote the sum of all connected diagrams contained in S'_n. It can be written in the form

$$S_n = S'_n - S''_n, \tag{9}$$

where S''_n is the sum of the corresponding nonconnected diagrams. The quantity S''_n can be represented as a sum, each term of which is the sum of all nonconnected diagrams with the fixed partition of the initial diagram into connected parts. Each such term is the product of several quantities $S_{\tilde{n}}$ with $\tilde{n} < n$. We can now perform expansion (9) for each $S_{\tilde{n}}$ and repeat this reasoning until all the obtained quantities of type $S_{\tilde{n}}$ coincide with S_1. Taking into account that $S_1 = S'_1$ (because there are no nonconnected diagrams in the first order in PT), we conclude that the quantity S_n can be represented as a finite sum of products of the quantities S'_j with $j \leq n$:

$$S_n = \sum \prod_k S'_{j_k}, \tag{10}$$

$$\sum_k j_k = n. \tag{11}$$

We now investigate the surface UV divergence of the quantity S_n (in the limit of removing the intermediate regularization). The surface UV divergence is understood to be the degree of divergence determined by the UV divergence index of a diagram without taking possible subdiagram divergences into account. When evaluating this index, we assume that all internal momenta of the diagram tend to infinity. In the coordinate space, this corresponds to the case where the coordinates of all the vertices tend to each other. If we then obtain a pole of order r_n under the integral sign, then the condition for the absence of the surface divergence for the quantity S_n is

$$2(n-1) - r_n > 0, \tag{12}$$

where we take $n-1$ two-dimensional volume elements into account and leave one volume element aside because of the translational invariance.

To estimate the pole order r_n, we expand the quantity S_n using formula (10), assuming this formula to be applied to expressions under the integral sign, while we take all integration signs outside the summation sign. We now find the order r'_j of the pole that appears when all the coordinates of all the vertices in the quantity S'_j tend to each other. It follows easily from the form (6) of the propagator that

$$\Delta(x) \sim \log \frac{1}{x^2}, \qquad \text{at} \quad x \to 0. \tag{13}$$

Therefore each superpropagator joining vertices of different types produces a second-order pole when the coordinates of these two vertices tend to each other (see (7)),

$$e^{\Delta(x)} \sim \frac{1}{x^2}, \qquad \text{at} \quad x \to 0, \tag{14}$$

while each superpropagator joining vertices of the same type produces a second-order zero (see (8)),

$$e^{-\Delta(x)} \sim x^2, \qquad \text{at} \quad x \to 0. \tag{15}$$

We now represent the integrand in the quantity S'_j in terms of the superpropagators, as described after formula (8). Because we join each pair of vertices by a superpropagator of one or the other type, we can use asymptotic expressions (14) and (15) to find easily the total asymptotic expression in the case where all the coordinates of all the vertices tend to each other, i.e., to find r'_j.

Let S'_j contain l_1 vertices of the first type and l_2 vertices of the second type, $l_1 + l_2 = j$. Then the respective numbers of propagators connecting vertices of different and the same types are

$$l_1 l_2 \qquad \text{and} \qquad \frac{l_1(l_1-1)}{2} + \frac{l_2(l_2-1)}{2}. \tag{16}$$

The overall order of the pole is therefore

$$r'_j = 2\left(l_1 l_2 - \frac{l_1(l_1-1)}{2} - \frac{l_2(l_2-1)}{2}\right) = j - (l_1 - l_2)^2 \leq j. \tag{17}$$

Formula (10) implies that

$$r_n = \max \sum_k r'_{j_k}, \tag{18}$$

where the maximum is taken over all summands (10). Hence, using estimate (17) and condition (11), we find that

$$r_n \leq \max \sum_k j_k = n. \tag{19}$$

Using this inequality, we can estimate the left-hand side of condition (12) for the absence of the surface divergence for the quantity S_n and thus obtain

$$2(n-1) - r_n \geq 2(n-1) - n = n - 2, \tag{20}$$

which means that such a divergence is absent for $n > 2$. The UV divergence is clearly absent for $n = 1$ (because of the normal-ordering symbol in Lagrangian (3), which prohibits loops containing only one vertex), and the only case when the surface divergence exists in the sum of all connected diagrams of order n in γ is the case $n = 2$.

3. Analyzing divergences in the second order in γ

At $n = 2$, condition (12) is broken only if $r_2 \geq 2$. By virtue of formula (18), this can happen only if $r'_2 \geq 2$ because $r'_1 = 0$ (no loops containing only one vertex permitted), and condition (11) must be satisfied. Formula (17) then tells us that this is possible only if $l_1 = l_2$; then $r'_2 = 2$ and hence $r_2 = 2$. We therefore conclude that the surface UV divergence can occur only for the sum of all connected diagrams of the second order in γ with vertices of different types ($l_1 = l_2 = 1$) and with the fixed arrangement for joining external lines to the vertices. This divergence turns out to be logarithmic because the left-hand side of condition (12) then vanishes.

The UV-divergent sum of the diagrams just described contributes to any Green's function in the second order of the PT in γ. We now investigate the presence of the surface UV divergence in the case of the sum of all such contributions. For the N-point Green's function ($N > 0$), the sum of all contributions of the second order in γ that correspond to the fixed type of vertices (in the case of vertices of different types) can be found from Lagrangian (3) by virtue of formulas (4)-(8):

$$G_N^{(2)}(y_1, \ldots, y_N) = -\left(\frac{\gamma}{2}\right)^2 \int d^2x_1 d^2x_2 \prod_{i=1}^{N} \left(\sum_{k_i=1,2} i\,(-1)^{k_i+1} \Delta(y_i - x_{k_i}) \right) e^{\Delta(x_1 - x_2)}. \tag{21}$$

where the summation ranges over all possible variants of joining external lines to various internal vertices. As $(x_1 - x_2) \to 0$, the expression in brackets in (21) under the product sign behaves as

$$\sum_{k_i=1,2} i\,(-1)^{k_i+1} \Delta(y_i - x_{k_i}) = i\Big(\Delta(y_i - x_1) - \Delta(y_i - x_2)\Big) \sim O(x_1 - x_2), \tag{22}$$

i.e., it improves the convergence of the integral at the point $(x_1 - x_2) = 0$. Because we have just proved that without taking this into account, the integration in the vicinity of this point

(i.e., when the coordinates of the vertices tend to each other) can develop at most a logarithmic UV singularity, we conclude that with asymptotic behavior (22) now taken into account, the N-point Green's function in the second order of the PT in γ does not contain a surface UV divergence for $N > 0$.

Because, as already mentioned, no UV divergences occur in the first order of the PT in γ, the sums of diagrams of the Green's functions cannot contain divergent subdiagrams in the second order of the PT. Therefore, because the surface (i.e., not taking possible subdiagram divergences into account) divergence is absent, the N-point Green's function is actually UV finite in the second order of the PT in γ for $N > 0$.

4. The absence of subdiagram divergences

We now address the question whether the quantities S_n with $n > 2$ can contain UV divergences because of the presence of UV divergent sums of subdiagrams.

At $n = 3$, only a sum of subdiagrams of the second order in γ with vertices of different types can be UV divergent (see the beginning of Sec. 3). Only connected diagrams enter the quantity S_n, and each subdiagram of the second order therefore contains at least one line that is external with respect to this subdiagram. In the total sum of all the diagrams constituting the quantity , we encounter the summation over all possible ways of joining this line to the subdiagram vertices, which is completely analogous to the summation over all possible ways of joining external lines to a diagram of the second order (see Sec. 3). Using formulas analogous to (21) and (22), we can then conclude that the sum of subdiagrams of the second order in γ is UV finite. Hence, there is no UV divergence for the quantity S_n in the case $n = 3$.

This, in particular, means that only a sum of subdiagrams of the second order might result in an UV divergence in the case $n = 4$, and we can repeat the above reasoning for this case as well. Further, using the induction process, we obtain the result: the sum of all connected diagrams of order $n > 2$ in the PT in γ is actually (not only by counting UV-divergence index) UV finite. This result holds for diagrams with any number of external lines and, in particular, for the vacuum diagrams with $n > 2$.

Combining this result with the conclusions in Sec. 3, we obtain the actual UV finiteness of all connected diagrams of the N-point Green's function in all orders of the PT in γ for $N > 0$ (i.e., for all diagrams except vacuum diagrams). Hence, we obtain the UV finiteness of all the Green's functions without vacuum loops because the diagrams for such Green's functions are products of connected nonvacuum diagrams.

5. Conclusion

For the PT in γ (and hence in the fermion mass M) in bosonic formulation (3) of 2D-QED, we have proved the UV finiteness of the following quantities:

1. For terms of order $n > 2$ of the PT, the sum of all connected diagrams with any (in particular, zero) number of external lines is UV finite; this remains valid if we fix the way the external lines of a diagram are attached to its vertices.

2. In the second order of the PT, all nonvacuum Green's functions, i.e., the sums of all diagrams with a nonzero number of external lines, become UV finite after summing up all the contributions corresponding to all possible ways of joining the external lines of the diagram to its vertices.

In particular, we then conclude that Green's functions without vacuum loops are UV finite in all orders of the PT in γ.

Only the sum of all vacuum diagrams and the sum of all diagrams with a nonzero number of external lines but with a fixed way of joining these lines to the vertices of the diagram are UV divergent in the second order of the PT in γ. This divergence is logarithmic. The vacuum Green's function (i. e. without external lines) is therefore logarithmically UV divergent, but only in the second order of the PT in γ.

Because only Green's functions without vacuum loops are important, we have thus proved the UV finiteness of the chiral PT for 2D-QED.

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